

On composition of Segal-Bargmann transforms

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Abstract. We introduce and discuss some basic properties of a natural integral transform mapping isometrically the standard Hilbert space on the real line into the two-dimensional Bargmann-Fock space. This transform looks like the well-known one-dimensional Segal-Bargmann transform. Its left inverse is also obtained. Generalization to high dimensions is possible for $d = 2^n$. Once connected to some well-known operators namely the Fourier transform and the Weyl operator we can recuperate isometrically the two-dimensional Bargmann-Fock space from its analogue on the complex plane.

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1. Introduction

The well-known Segal-Bargmann transform ([2, 6, 13, 7, 11]),

$$\mathcal{B}_d[\psi](z) = \left(\frac{\nu}{\pi}\right)^{\frac{3d}{4}} \int_{\mathbb{R}^d} e^{-\frac{\nu}{2}(z^2+x^2)+\nu\sqrt{2}z \cdot x} \psi(x) dx; \quad z \in \mathbb{C}^d, \nu > 0,$$

made the quantum mechanical configuration space $L^2(\mathbb{R}^d; dx)$ unitary isomorphic to the phase space consisting of all square integrable holomorphic functions with respect to the gaussian measure $e^{-\nu|z|^2} d\lambda$ on the d -dimensional complex space, $d\lambda$ being the Lebesgue measure on \mathbb{C}^d . Using the one and the two-dimensional Segal-Bargmann transforms \mathcal{B}_1 and \mathcal{B}_2 , and the multiplication operator $\mathcal{M}_\nu \psi := e^{-\frac{\nu}{2}|z|^2} \psi$, a natural unitary integral transform mapping the standard Hilbert space $L^2(\mathbb{R}; dx)$ on the real line into the classical Bargmann-Fock space $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$ on the two-dimensional complex space can be constructed, to wit

$$\mathcal{G}^\nu := \mathcal{B}_2 \circ \mathcal{M}_\nu \circ \mathcal{B}_1.$$

The main purpose of the present paper is to show that \mathcal{G}^ν can be reduced further to an extremely integral operator that looks like the one-dimensional Segal-Bargmann transform \mathcal{B}_1 . More precisely, we show that $\mathcal{G}^\nu \varphi = \left(\frac{\nu}{\pi}\right)^{\frac{1}{2}} \mathcal{C}_\psi(\mathcal{B}_1 \varphi)$ for every given $\varphi \in L^2(\mathbb{R}; dx)$,

where $\mathcal{C}_\psi f = f \circ \psi$ is the composition operator with the specific symbol

$$\psi(z_1, z_2) = \frac{z_1 + iz_2}{\sqrt{2}}.$$

This fact can be generalized to high dimensions when $d = 2^n$ by considering the transform

$$\mathcal{G}_n^\nu := \mathcal{B}_{2^n} \mathcal{M}_{2^{n-1}} \mathcal{G}_{n-1}^\nu \quad \text{for } n > 1 \text{ and } \quad \mathcal{G}_1^\nu := \mathcal{G}^\nu \quad \text{when } n = 1,$$

where \mathcal{M}_k is the multiplication operator defined by

$$\mathcal{M}_k \psi(z) := e^{-\frac{\nu}{2}|z|^2} \psi(z); \quad z = (z_1, \dots, z_k) \in \mathbb{C}^k.$$

As interesting applications, we investigate further properties of the transform \mathcal{G}^ν when combined with the well-known Fourier transform and Weyl operator. As a consequence, we introduce two integral operators connecting the one-dimensional Bargmann-Fock space $\mathcal{F}^{2,\nu}(\mathbb{C})$ to the two-dimensional one $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$.

To present these ideas, we adopt the following structure: in Section 2, we review the basic backgrounds related to the classical Bargmann-Fock space and Segal-Bargmann transform. In Section 3, we explain the main idea of construction based on composing the one and the two-dimensional Segal-Bargmann transforms. The left inverse of this transform is also studied. Section 4 is devoted to the study of two operators related to the Fourier transform and the Weyl operator. We conclude with some remarks, in particular we will see how this approach can be generalized on 2^n -dimensions and guess a generalized formula for the symbol in that case.

2. Notation and preliminaries

Let $L^2(\mathbb{R}^d; dx)$ denote the classical Hilbert space on the d -real space \mathbb{R}^d with respect to its standard Lebesgue measure $dx = dx_1 \cdots dx_n$. An orthogonal basis of $L^2(\mathbb{R}^n; dx)$ is given by the multi-dimensional Hermite functions

$$h_m^\nu(x) := (-1)^{|m|} e^{\frac{\nu}{2}x^2} \frac{\partial^{|m|}}{\partial x^{m_1} \cdots \partial x^{m_n}} \left(e^{-\nu x^2} \right) = \prod_{\ell=1}^d h_{m_\ell}^\nu(x_\ell), \quad (2.1)$$

for varying $m = (m_1, \dots, m_d) \in (\mathbb{Z}^+)^d$, where $h_{m_\ell}^\nu(x_\ell)$ is the one-dimensional Hermite function (see [5, 8]). Their norm is known to be given explicitly by

$$\|h_m^\nu\|_{L^2(\mathbb{R}^d; dx)}^2 = 2^{|m|} \nu^{|m|} m! \left(\frac{\pi}{\nu} \right)^{d/2}. \quad (2.2)$$

Here, we have used the multi-index notation $|m| = m_1 + \cdots + m_d$ and $m! = m_1! \cdots m_d!$.

The classical Bargmann-Fock space $\mathcal{F}^{2,\nu}(\mathbb{C}^d)$; $\nu > 0$, on the d -dimensional complex space \mathbb{C}^d is defined to be ([2, 13, 11])

$$\mathcal{F}^{2,\nu}(\mathbb{C}^d) = \text{Hol}(\mathbb{C}^d) \cap L^2(\mathbb{C}^d; e^{-\nu|z|^2} d\lambda),$$

where $\text{Hol}(\mathbb{C}^d)$ denotes the space of holomorphic functions and $d\lambda$ the Lebesgue measure on \mathbb{C}^d .

It turns out that the reproducing kernel Hilbert space $\mathcal{F}^{2,\nu}(\mathbb{C}^d)$ can be realized as the range of $L^2(\mathbb{R}^d; dx)$ by the special integral (Segal-Bargmann) transform

$$\mathcal{B}_d[\psi](z) = \left(\frac{\nu}{\pi} \right)^{\frac{3d}{4}} \int_{\mathbb{R}^d} e^{-\frac{\nu}{2}(z^2 + x^2) + \nu\sqrt{2}z \cdot x} \psi(x) dx, \quad (2.3)$$

where $z \cdot w = \sum_{k=1}^d z_k w_k$ for given $z = (z_1, \dots, z_d), w = (w_1, \dots, w_d) \in \mathbb{C}^d$.

This transform \mathcal{B}_d is a unitary integral transform from $L^2(\mathbb{R}^d; dx)$ onto $\mathcal{F}^{2,\nu}(\mathbb{C}^d)$ and maps h_m^ν to the standard orthogonal basis of $\mathcal{F}^{2,\nu}(\mathbb{C}^d)$, constituted from the monomials. More exactly, we have [2]

$$[\mathcal{B}_d h_m^\nu](z) = \left(\frac{\nu}{\pi}\right)^{\frac{d}{4}} 2^{\frac{|m|}{2}} \nu^{|m|} z^m. \quad (2.4)$$

Notice that the kernel function of \mathcal{B}_d given by the exponential function

$$A_d^\nu(z; x) := e^{-\frac{\nu}{2}(z^2 + x^2) + \nu\sqrt{2}z \cdot x}, \quad (2.5)$$

can be seen as the generating function of the multi-dimensional Hermite functions $h_m^\nu(x)$.

Associated to \mathcal{B}_1 , \mathcal{B}_2 and the multiplication operator

$$\mathcal{M}_\nu \psi(z) := e^{-\frac{\nu}{2}|z|^2} \psi(z),$$

we define the integral transform \mathcal{G}^ν on $L^2(\mathbb{R}; dx)$ to be

$$\mathcal{G}^\nu := \mathcal{B}_2 \circ \mathcal{M}_\nu \circ \mathcal{B}_1.$$

Then, it is clear that $\mathcal{B}_2 \circ \mathcal{M}_\nu \circ \mathcal{B}_1[\varphi]$ is an holomorphic function on \mathbb{C}^2 belonging to the Hilbert space $L^2(\mathbb{C}^2; e^{-\nu(|z_1|^2 + |z_2|^2)} d\lambda)$ for every given $\varphi \in L^2(\mathbb{R})$. This follows since $\mathcal{M}_\nu \circ \mathcal{B}_1[\varphi]$ is a square integrable function on \mathbb{R}^2 and \mathcal{B}_2 is the classical Segal-Bargmann transform from $L^2(\mathbb{R}^2; dxdy)$ onto $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$. Moreover, by means of the isometries \mathcal{B}_2 , \mathcal{M}_ν and \mathcal{B}_1 , we can check easily the following

Proposition 2.1. *The integral transform $\mathcal{G}^\nu := \mathcal{B}_2 \circ \mathcal{M}_\nu \circ \mathcal{B}_1$ defines an isometric operator from $L^2(\mathbb{R}; dx)$ into $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$.*

Proof. We only need to prove $\|\mathcal{G}^\nu[\varphi]\|_{\mathcal{F}^{2,\nu}(\mathbb{C}^2)} = \|\varphi\|_{L^2(\mathbb{R}; dx)}$ for every $\varphi \in L^2(\mathbb{R}; dx)$. Indeed, we have

$$\|\mathcal{B}_2 \circ \mathcal{M}_\nu \circ \mathcal{B}_1[\varphi]\|_{\mathcal{F}^{2,\nu}(\mathbb{C}^2)} = \|\mathcal{M}_\nu \circ \mathcal{B}_1[\varphi]\|_{L^2(\mathbb{R}^2; dxdy)} = \|\mathcal{B}_1[\varphi]\|_{\mathcal{F}^{2,\nu}(\mathbb{C})} = \|\varphi\|_{L^2(\mathbb{R}; dx)}.$$

3. Main results related to the operator $\mathcal{B}_2 \circ \mathcal{M}_\nu \circ \mathcal{B}_1$.

The first main result of this paper is

Theorem 3.1. *For every given $\varphi \in L^2(\mathbb{R}; dx)$ and $(z_1, z_2) \in \mathbb{C}^2$, we have*

$$\mathcal{G}^\nu[\varphi](z_1, z_2) := [\mathcal{B}_2 \circ \mathcal{M}_\nu \circ \mathcal{B}_1(\varphi)](z_1, z_2) = \left(\frac{\nu}{\pi}\right)^{\frac{1}{2}} \mathcal{B}_1[\varphi]\left(\frac{z_1 + iz_2}{\sqrt{2}}\right). \quad (3.1)$$

In order to prove it, we begin with the following

Lemma 3.2. *For every fixed $(z_1, z_2) \in \mathbb{C}^2$ and $x \in \mathbb{R}$, we set*

$$L(x; (z_1, z_2)) := \int_{\mathbb{R}^2} A_1(t_1 + it_2; x) A_2((z_1, z_2); (t_1, t_2)) e^{-\frac{\nu}{2}(t_1^2 + t_2^2)} dt_1 dt_2.$$

Then, we have

$$L(x; (z_1, z_2)) = \left(\frac{\pi}{\nu}\right) A_1\left(\frac{z_1 + iz_2}{\sqrt{2}}; x\right).$$

Proof. Starting from the explicit expression (2.5) of the kernel functions A_d for $d = 1$ and $d = 2$, and making appeal of the Fubini's theorem as well as the explicit formula for the gaussian integral

$$\int_{\mathbb{R}} e^{-ax^2+bx} dx = \left(\frac{\pi}{a}\right)^{\frac{1}{2}} e^{\frac{b^2}{4a}}, \quad (3.2)$$

we get

$$\begin{aligned} L(x; (z_1, z_2)) &= \left(\frac{2\pi}{\nu}\right)^{1/2} e^{-\frac{\nu}{2}(3x^2+z_1^2-z_2^2-4ixz_2)} \int_{\mathbb{R}} e^{-2\nu t_1^2 + \sqrt{2}\nu((x+z_1)+(x-iz_2))t_1} dt_1 \\ &= \left(\frac{\pi}{\nu}\right) e^{-\frac{\nu}{2}(3x^2+z_1^2-z_2^2-4ixz_2)} e^{\frac{\nu}{4}((x+z_1)+(x-iz_2))^2} \\ &= \left(\frac{\pi}{\nu}\right) e^{-\frac{\nu}{2}(x^2+\frac{1}{2}(z_1^2-z_2^2+2iz_1z_2))+\nu x(z_1+iz_2)} \\ &= \left(\frac{\pi}{\nu}\right) e^{-\frac{\nu}{2}(w^2+x^2)+\nu\sqrt{2}w.x}, \end{aligned}$$

where w stands for $w = \frac{z_1 + iz_2}{\sqrt{2}}$. This completes the proof. ■

Subsequently, we can proceed to the proof of Theorem 3.1.

Proof of Theorem 3.1. Set

$$\begin{aligned} \phi(t_1, t_2) &:= \phi(z) := \mathcal{M}_\nu \mathcal{B}_1 \varphi(z) \\ &= \left(\frac{\nu}{\pi}\right)^{\frac{3}{4}} e^{-\frac{\nu}{2}|z|^2} \int_{\mathbb{R}} A_1(z; x) \varphi(x) dx \end{aligned}$$

for every given $z = t_1 + it_2 \in \mathbb{C}$ with $(t_1, t_2) \in \mathbb{R}^2$. Then, we can rewrite the operator $\mathcal{G}^\nu = \mathcal{B}_2 \circ \mathcal{M}_\nu \circ \mathcal{B}_1$ as

$$\begin{aligned} \mathcal{G}^\nu[\varphi](z_1, z_2) &= \mathcal{B}_2[\phi](z_1, z_2) \\ &= \left(\frac{\nu}{\pi}\right)^{\frac{6}{4}} \int_{\mathbb{R}^2} A_2((z_1, z_2); (t_1, t_2)) \phi(t_1, t_2) dt_1 dt_2 \\ &= \left(\frac{\nu}{\pi}\right)^{\frac{9}{4}} \int_{\mathbb{R}} L(x; (z_1, z_2)) \varphi(x) dx \end{aligned}$$

Therefore, by means of Lemma 3.2, we obtain

$$\begin{aligned} \mathcal{G}^\nu[\varphi](z_1, z_2) &= \left(\frac{\nu}{\pi}\right)^{\frac{1}{2}} \left(\frac{\nu}{\pi}\right)^{\frac{3}{4}} \int_{\mathbb{R}} A_1\left(\frac{z_1 + iz_2}{\sqrt{2}}; x\right) \varphi(x) dx \\ &= \left(\frac{\nu}{\pi}\right)^{\frac{1}{2}} \mathcal{B}_1[\varphi]\left(\frac{z_1 + iz_2}{\sqrt{2}}\right). \end{aligned}$$

This ends the proof. ■

As immediate consequence, we have

Corollary 3.3. *Keep notations as above and set $\varphi_m^\nu(\xi) := \xi^m e^{-\frac{\nu}{2}|\xi|^2}$. Then, we have*

$$\mathcal{G}^\nu[h_m](z_1, z_2) = \left(\frac{\nu}{\pi}\right)^{\frac{3}{4}} \nu^m (z_1 + iz_2)^m$$

and

$$\mathcal{B}_2(\varphi_m^\nu)(z_1, z_2) = \left(\frac{\nu}{\pi}\right)^{\frac{1}{2}} \left(\frac{z_1 + iz_2}{\sqrt{2}}\right)^m.$$

Proof. The proof of the first assertion can be handled making use of Theorem 3.1. Indeed,

$$\begin{aligned} \mathcal{B}_2 \circ \mathcal{M}_\nu \circ \mathcal{B}_1[h_m^\nu](z_1, z_2) &= \left(\frac{\nu}{\pi}\right)^{\frac{1}{2}} \mathcal{B}_1[h_m^\nu] \left(\frac{z_1 + iz_2}{\sqrt{2}}\right) \\ &= \left(\frac{\nu}{\pi}\right)^{\frac{1}{2}} \left(\frac{\nu}{\pi}\right)^{\frac{1}{4}} 2^{\frac{m}{2}} \nu^m \left(\frac{z_1 + iz_2}{\sqrt{2}}\right)^m \end{aligned}$$

The second one can be checked by comparing the previous identity to the following

$$\begin{aligned} \mathcal{B}_2 \circ \mathcal{M}_\nu \circ \mathcal{B}_1[h_m^\nu](z_1, z_2) &= \mathcal{B}_2 \left(\xi \mapsto e^{-\frac{\nu}{2}|\xi|^2} \mathcal{B}_1[h_m^\nu](\xi) \right) (z_1, z_2) \\ &= \left(\frac{\nu}{\pi}\right)^{\frac{1}{4}} 2^{\frac{m}{2}} \nu^m \mathcal{B}_2 \left(\xi \mapsto e^{-\frac{\nu}{2}|\xi|^2} \xi^m \right) (z_1, z_2), \end{aligned}$$

that can be checked easily using (2.4). ■

Remark 3.4. The restriction of $\mathcal{B}_2 \circ \mathcal{M}_\nu \circ \mathcal{B}_1[\varphi]$ to $\mathbb{C} \times \{0\}$ coincides with the classical Segal-Bargmann transform $\left(\frac{\nu}{\pi}\right)^{\frac{1}{2}} \mathcal{B}_1[\varphi](z_1/\sqrt{2})$.

The transform defined from $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$ to $L^2(\mathbb{R}; dx)$ by

$$\mathcal{R}^\nu := \mathcal{B}_1^{-1} \circ Proj \circ M_{-\nu} \circ \mathcal{B}_2^{-1}$$

is the left inverse of $\mathcal{G}^\nu := \mathcal{B}_2 \circ \mathcal{M}_\nu \circ \mathcal{B}_1$. Here, *Proj* stands for the orthogonal projection operator from $L^2(\mathbb{C}; e^{-\nu|\xi|^2} d\lambda)$ to the classical Bargmann-Fock space. Indeed, It is easy to check that for $\phi \in L^2(\mathbb{R}; dx)$ we have $\mathcal{R}^\nu \mathcal{G}^\nu \phi = \mathcal{B}_1^{-1} Proj(\mathcal{B}_1 \phi)$. However, $\mathcal{B}_1 \phi$ is belonging to the classical Bargmann space then $Proj(\mathcal{B}_1 \phi) = \mathcal{B}_1 \phi$. This leads to $\mathcal{R}^\nu \mathcal{G}^\nu = Id$ and therefore \mathcal{R}^ν is the left inverse of \mathcal{G}^ν .

In what follows we give the representation of the operator \mathcal{R}^ν in terms of the inverse of \mathcal{B}_1 and a composition operator with specific symbol ψ_2 . To this end, let us recall that for $\phi \in L^2(\mathbb{R}; dx)$,

$$\mathcal{G}^\nu \phi(z_1, z_2) := \left(\frac{\nu}{\pi}\right)^{\frac{1}{2}} \mathcal{B}_1[\phi] \left(\frac{z_1 + iz_2}{\sqrt{2}}\right) = \left(\frac{\nu}{\pi}\right)^{\frac{1}{2}} \mathcal{C}_\psi(\mathcal{B}_1 \phi)(z_1, z_2).$$

Namely,

Theorem 3.5. *For every $f \in \mathcal{F}^{2,\nu}(\mathbb{C}^2)$, we have*

$$\mathcal{R}^\nu f = \left(\frac{\nu}{\pi}\right)^{-\frac{1}{2}} \mathcal{B}_1^{-1}[\mathcal{C}_{\psi_2}(f)],$$

where the symbol $\psi_2 : \mathbb{C} \rightarrow \mathbb{C}^2$ is given by $\psi_2(\xi) := \left(\frac{\xi}{\sqrt{2}}, -i\frac{\xi}{\sqrt{2}}\right)$. Explicitly,

$$\mathcal{R}^\nu f(x) = \left(\frac{\nu}{\pi}\right)^{\frac{1}{4}} \int_{\mathbb{C}} \overline{A_1(\xi, x)} (f \circ \psi_2)(\xi) e^{-\nu|\xi|^2} d\lambda(\xi).$$

To prove this theorem, let $f \in \mathcal{F}^{2,\nu}(\mathbb{C}^2)$ and set

$$h(w) := M_{-\nu} \mathcal{B}_2^{-1} f(t_1, t_2) \text{ where } w = t_1 + it_2.$$

Then, we have

$$h(w) = \left(\frac{\nu}{\pi}\right)^{\frac{6}{4}} \int_{\mathbb{C}^2} e^{-\frac{\nu}{2}(\bar{z}_1^2 + \bar{z}_2^2) + \nu\sqrt{2}(\bar{z}_1 t_1 + \bar{z}_2 t_2)} f(z_1, z_2) e^{-\nu(|z_1|^2 + |z_2|^2)} d\lambda(z_1, z_2)$$

Now, we use the formula of the orthogonal projection for the Bargmann-Fock space to prove the following lemma and then the theorem will be a direct consequence.

Lemma 3.6.

$$Proj(h)(\xi) = \left(\frac{\nu}{\pi}\right)^{-\frac{1}{2}} f\left(\frac{\xi}{\sqrt{2}}, -i\frac{\xi}{\sqrt{2}}\right) \forall \xi \in \mathbb{C}.$$

Proof. It is a well-known fact that the formula of the orthogonal projection for the Bargmann-Fock space is expressed in term of the reproducing kernel see for example [13]. Explicitly, for a function g in $L^2(\mathbb{C}; e^{-\nu|z|^2} d\lambda)$ we have

$$Proj(g)(\xi) := \int_{\mathbb{C}} K(\xi, w) g(w) e^{-\nu|w|^2} d\lambda(w) = \frac{\nu}{\pi} \int_{\mathbb{C}} e^{\nu\xi\bar{w}} g(w) e^{-\nu|w|^2} d\lambda(w).$$

In particular, for a fixed ξ . We apply the last formula for $g = h$ and thus thanks to Fubini's theorem we obtain

$$Proj h(\xi) = \left(\frac{\nu}{\pi}\right)^{\frac{10}{4}} \int_{\mathbb{C}^2} e^{-\frac{\nu}{2}(\bar{z}_1^2 + \bar{z}_2^2)} f(z_1, z_2) I(\xi, \bar{z}_1, \bar{z}_2) e^{-\nu(|z_1|^2 + |z_2|^2)} d\lambda(z_1, z_2) \quad (3.3)$$

Where we have set

$$I(\xi, \bar{z}_1, \bar{z}_2) = \int_{\mathbb{R}^2} e^{\nu\xi t_1 - \nu\xi t_2 i + \nu\sqrt{2}\bar{z}_1 t_1 + \nu\sqrt{2}\bar{z}_2 t_2} e^{-\nu(t_1^2 + t_2^2)} d\lambda(t_1, t_2)$$

thanks to Fubini's theorem combined with the Gaussian integral we get

$$I = \frac{\pi}{\nu} e^{\frac{\nu}{2}(\bar{z}_1^2 + \bar{z}_2^2) + \nu\xi \frac{(\bar{z}_1 - i\bar{z}_2)}{\sqrt{2}}}.$$

By replacing $I(\xi, \bar{z}_1, \bar{z}_2)$ in (3) by its expression, it follows that

$$\begin{aligned} Proj(h)(\xi) &= \left(\frac{\nu}{\pi}\right)^{\frac{6}{4}} \int_{\mathbb{C}^2} e^{\nu\xi \frac{(\bar{z}_1 - i\bar{z}_2)}{\sqrt{2}}} f(z_1, z_2) e^{-\nu|(z_1, z_2)|^2} d\lambda(z_1, z_2) \\ &= \left(\frac{\nu}{\pi}\right)^{-\frac{2}{4}} \int_{\mathbb{C}^2} \left(\frac{\nu}{\pi}\right)^2 e^{\nu(\frac{\xi}{\sqrt{2}}\bar{z}_1 - \frac{i\xi}{\sqrt{2}}\bar{z}_2)} f(z_1, z_2) e^{-\nu|(z_1, z_2)|^2} d\lambda(z_1, z_2). \end{aligned}$$

However, from the formula of the reproducing kernel of the two dimension Bargmann-Fock space we have that

$$\overline{K_{\nu}\left(\frac{\xi}{\sqrt{2}}, -\frac{\xi i}{\sqrt{2}}; z_1, z_2\right)} = \left(\frac{\nu}{\pi}\right)^2 e^{\nu(\frac{\xi}{\sqrt{2}}\bar{z}_1 - \frac{i\xi}{\sqrt{2}}\bar{z}_2)}$$

It follows from the reproducing kernel property that

$$Proj(h)(\xi) = \left(\frac{\nu}{\pi}\right)^{-\frac{1}{2}} f\left(\frac{\xi}{\sqrt{2}}, -\frac{i\xi}{\sqrt{2}}\right)$$

Remark 3.7. 1. Notice, that we have $\psi_1 \circ \psi_2 = id_{\mathbb{C}}$ but unfortunately $\psi_2 \circ \psi_1 \neq id_{\mathbb{C}^2}$.
 2. The transform \mathcal{G}^{ν} is not a coherent state transform in the sense that its kernel function can not be recovered as a bilateral generating function of the orthonormal bases of the Hilbert spaces $L^2(\mathbb{R}; dx)$ and $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$.

4. On some integral operators connecting the Bargmann-Fock spaces $\mathcal{F}^{2,\nu}(\mathbb{C})$ and $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$

In this section, we investigate further properties of the integral transform \mathcal{G}^ν when combined with some classical transforms. Namely, we will deal with the Fourier transform and the Weyl operator that are useful tools, especially in signal processing and quantum mechanics. [7, 13]

4.1. The operator $\mathcal{S}^\nu = \mathcal{G}^\nu \mathcal{F}^\nu \mathcal{B}_1^{-1}$

The first main result of this section is related to the Fourier transform defined here by

$$\mathcal{F}^\nu(f)(\xi) := \sqrt{\frac{\nu}{2\pi}} \int_{\mathbb{R}} f(x) e^{-\nu i x \xi} dx$$

acting on $L^2(\mathbb{R}; dx)$ as a bounded linear operator. Thanks to the well-known Plancherel-theorem, it turns out that the Fourier transform maps unitary $L^2(\mathbb{R}; dx)$ onto itself. Associated to \mathcal{F}^ν , we consider the transform

$$\mathcal{S}^\nu := \mathcal{G}^\nu \mathcal{F}^\nu \mathcal{B}_1^{-1}.$$

Clearly \mathcal{S}^ν maps isometrically $\mathcal{F}^{2,\nu}(\mathbb{C})$ into $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$. The explicit formula of \mathcal{S}^ν acting on $\mathcal{F}^{2,\nu}(\mathbb{C})$ is given by:

Theorem 4.1. *For a given $f \in \mathcal{F}^{2,\nu}(\mathbb{C})$, we have*

$$\mathcal{S}^\nu(f) = \left(\frac{\nu}{\pi}\right)^{-\frac{1}{4}} \mathcal{C}_{-i\psi}(f),$$

More precisely, we have

$$\mathcal{S}^\nu(f)(z_1, z_2) = \left(\frac{\nu}{\pi}\right)^{-\frac{1}{4}} f\left(\frac{-i(z_1 + iz_2)}{\sqrt{2}}\right)$$

for every $(z_1, z_2) \in \mathbb{C}^2$.

The proof of Theorem 4.1 lies essentially on the following well established fact, giving the explicit expression of the action of the Fourier transform \mathcal{F} on the inverse of Segal-Bargmann transform defined on $\mathcal{F}^{2,\nu}(\mathbb{C})$ by ([2])

$$\mathcal{B}_1^{-1}[f](x) := \left(\frac{\nu}{\pi}\right)^{\frac{3}{4}} \int_{\mathbb{C}} e^{-\frac{\nu}{2}(\bar{z}^2 + x^2) + \sqrt{2}\nu \bar{z}x} f(z) e^{-\nu|z|^2} d\lambda(z). \quad (4.1)$$

Namely, we assert the following

Proposition 4.2. *For every given function f belonging to the Bargmann-Fock space $\mathcal{F}^{2,\nu}(\mathbb{C})$, we have*

$$\mathcal{F}^\nu(\mathcal{B}_1^{-1}[f])(\xi) = -\mathcal{B}_1^{-1}[\tilde{f}](\xi),$$

where

$$\tilde{f}(z) := f(-iz).$$

Proof. Let $f \in \mathcal{F}^{2,\nu}(\mathbb{C})$. Then, we have

$$\mathcal{F}^\nu(\mathcal{B}_1^{-1}[f])(\xi) = \sqrt{\frac{\nu}{2\pi}} \left(\frac{\nu}{\pi}\right)^{\frac{3}{4}} \int_{\mathbb{C}} e^{-\frac{\nu}{2}\bar{z}^2 - \nu|z|^2} f(z) \left(\int_{\mathbb{R}} e^{-\frac{\nu}{2}x^2 + \nu x(\sqrt{2}\bar{z} - i\xi)} dx \right) d\lambda(z).$$

In the right hand side we recognize the Gaussian integral (3.2) with $a = \frac{\nu}{2}$ and $b = \nu(\sqrt{2}\bar{z} - i\xi)$. Therefore, it follows

$$\mathcal{F}^\nu(\mathcal{B}_1^{-1}[f])(\xi) = \left(\frac{\nu}{\pi}\right)^{\frac{3}{4}} \int_{\mathbb{C}} e^{-\frac{\nu}{2}(-\bar{z}^2 + \xi^2) + \nu\sqrt{2}(-i\bar{z})\xi} f(z) e^{-\nu|z|^2} d\lambda(z).$$

Making the change $z = -iw$ gives rise to

$$\begin{aligned} \mathcal{F}^\nu(\mathcal{B}_1^{-1}[f])(\xi) &= -\left(\frac{\nu}{\pi}\right)^{\frac{3}{4}} \int_{\mathbb{C}} e^{-\frac{\nu}{2}(\bar{w}^2 + \xi^2) + \nu\sqrt{2}\xi\bar{w}} f(-iw) e^{-\nu|w|^2} d\lambda(w) \\ &= -\mathcal{B}_1^{-1}[\tilde{f}](\xi). \end{aligned}$$

■

Proof of Theorem 4.1. It is immediate making use of Proposition 4.2 combined with Theorem 3.1. Indeed, we set $\psi(\xi) = \mathcal{F}^\nu(\mathcal{B}_1^{-1}f)(\xi)$ and then direct computations lead to

$$\mathcal{G}^\nu[\psi](z_1, z_2) = \left(\frac{\nu}{\pi}\right)^{-\frac{1}{4}} \int_{\mathbb{C}} \frac{\nu}{\pi} e^{\nu\bar{\xi}\left(-i\frac{(z_1 + iz_2)}{\sqrt{2}}\right)} f(\xi) e^{-\nu|\xi|^2} d\lambda(\xi).$$

Thence, the proof is completed making use of the reproducing kernel property. ■

4.2. On the operator $\mathcal{L}_a^\nu := \mathcal{G}^\nu \mathcal{T}_a^\nu \mathcal{B}_1^{-1}$

The Weyl operator is an important mathematical object in quantum physics see [7, 13]. It realizes a unitary operator from the Bargmann-Fock space $\mathcal{F}^{2,\nu}(\mathbb{C})$ onto itself. For given complex number a , the Weyl operator is defined by (see [13])

$$\begin{aligned} \mathcal{W}_a f(z) &:= e^{\nu(z\bar{a} - \frac{|a|^2}{2})} \mathcal{T}_a^\nu f(z) \\ &= e^{\nu(z\bar{a} - \frac{|a|^2}{2})} f(z - a) \\ &= k_a^\nu(z) f(z - a). \end{aligned}$$

for every $f \in \mathcal{F}^{2,\nu}(\mathbb{C})$ and $z \in \mathbb{C}$, where \mathcal{T}_a^ν denotes the well-known translation operator

$$\mathcal{T}_a^\nu f(z) := f(z - a)$$

and defines a unitary operator on the classical Hilbert space $L^2(\mathbb{R}; dx)$ when $a \in \mathbb{R}$ while $k_a^\nu(z) := \frac{K^\nu(a, z)}{\|K_a^\nu\|_{\mathcal{F}^{2,\nu}(\mathbb{C})}}$ is the normalized reproducing kernel of the Bargmann-Fock space.

Moreover, for fixed $a \in \mathbb{R}$, we consider the operator

$$\mathcal{L}_a^\nu := \mathcal{G}^\nu \mathcal{T}_a^\nu \mathcal{B}_1^{-1}$$

from $\mathcal{F}^{2,\nu}(\mathbb{C})$ the Bargmann-Fock space on \mathbb{C} into $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$ the Bargmann-Fock space on \mathbb{C}^2 . Obviously, \mathcal{L}_a^ν is an isometry. Its explicit expression is given in terms of the Weyl operator. Namely, we have the following

Theorem 4.3. *For every given $f \in \mathcal{F}^{2,\nu}(\mathbb{C})$ and $(z_1, z_2) \in \mathbb{C}^2$, we have*

$$\mathcal{L}_a^\nu f(z_1, z_2) = \left(\frac{\nu}{\pi}\right)^{-\frac{1}{4}} \mathcal{W}_{\frac{a}{\sqrt{2}}} f\left(\frac{z_1 + iz_2}{\sqrt{2}}\right).$$

Proof. For given $f \in \mathcal{F}^{2,\nu}(\mathbb{C})$, we set

$$h(x) := \mathcal{B}_1^{-1}f(x).$$

Then, it is easy to check that

$$\mathcal{T}_a^\nu h(x) = \left(\frac{\nu}{\pi}\right)^{\frac{3}{4}} e^{\nu(-\frac{x^2}{2} - \frac{a^2}{2} + xa)} \int_{\mathbb{C}} e^{\nu(-\frac{\bar{w}^2}{2} + \sqrt{2}\bar{w}x - \sqrt{2}\bar{w}a)} f(w) e^{-\nu|w|^2} d\lambda(w). \quad (4.2)$$

To compute $\mathcal{G}^\nu \mathcal{T}_a^\nu \mathcal{B}_1^{-1} f(z_1, z_2) = \mathcal{G}^\nu \mathcal{T}_a^\nu h(z_1, z_2)$, we start from (4.2) and use the fact that

$$A_1\left(\frac{z_1 + iz_2}{\sqrt{2}}, x\right) = e^{\nu\left(-\frac{z_1^2}{4} + \frac{z_2^2}{4} - \frac{iz_1 z_2}{2} - \frac{x^2}{2} + z_1 x + iz_2 x\right)},$$

combined with Fubini's theorem to get

$$\begin{aligned} \mathcal{G}^\nu \mathcal{T}_a^\nu \mathcal{B}_1^{-1} f(z_1, z_2) &= \left(\frac{\nu}{\pi}\right)^{\frac{3}{4}} \left(\frac{\nu}{\pi}\right)^{\frac{1}{2}} e^{\nu\left(-\frac{z_1^2}{4} + \frac{z_2^2}{4} - \frac{iz_1 z_2}{2} - \frac{a^2}{2}\right)} \\ &\quad \times \int_{\mathbb{R}} e^{\nu(-x^2 + x(z_1 + iz_2 + a))} \left(\int_{\mathbb{C}} e^{\nu\left(-\frac{\bar{w}^2}{2} + \sqrt{2}\bar{w}x - \sqrt{2}\bar{w}a\right)} f(w) e^{-\nu|w|^2} d\lambda(w) \right) dx \\ &= \left(\frac{\nu}{\pi}\right)^{\frac{3}{4}} \left(\frac{\nu}{\pi}\right)^{\frac{1}{2}} e^{\nu\left(-\frac{z_1^2}{4} + \frac{z_2^2}{4} - \frac{iz_1 z_2}{2} - \frac{a^2}{2}\right)} \\ &\quad \times \int_{\mathbb{C}} e^{\nu\left(-\frac{\bar{w}^2}{2} - \sqrt{2}\bar{w}a\right)} f(w) \left(\int_{\mathbb{R}} e^{\nu(-x^2 + x(z_1 + iz_2 + a + \sqrt{2}\bar{w}))} dx \right) e^{-\nu|w|^2} d\lambda(w). \end{aligned}$$

Straightforward computations making use of the Gaussian integral formula yields

$$\mathcal{G}^\nu \mathcal{T}_a^\nu \mathcal{B}_1^{-1} f(z_1, z_2) = \left(\frac{\nu}{\pi}\right)^{\frac{3}{4}} e^{\nu\left(\frac{a}{2}(z_1 + iz_2) - \frac{a^2}{4}\right)} \int_{\mathbb{C}} e^{\nu\left(\frac{z_1 + iz_2}{\sqrt{2}} - \frac{a}{\sqrt{2}}\right)\bar{w}} f(w) e^{-\nu|w|^2} d\lambda(w),$$

and therefore, we obtain

$$\mathcal{G}^\nu \mathcal{T}_a^\nu \mathcal{B}_1^{-1} f(z_1, z_2) = \left(\frac{\nu}{\pi}\right)^{-\frac{1}{4}} e^{\nu\left(-\frac{a^2}{4} + \frac{z_1 a}{2} + i\frac{z_2 a}{2}\right)} f\left(\frac{z_1 + iz_2}{\sqrt{2}} - \frac{a}{\sqrt{2}}\right),$$

thanks to the reproducing kernel property for the Bargmann-Fock space. Finally, it is not so difficult to realize that

$$\mathcal{G}^\nu \mathcal{T}_a^\nu \mathcal{B}_1^{-1} f(z_1, z_2) = \left(\frac{\nu}{\pi}\right)^{-\frac{1}{4}} \mathcal{W}_{\frac{a}{\sqrt{2}}} f\left(\frac{z_1 + iz_2}{\sqrt{2}}\right).$$

This achieves the proof. ■

Remark 4.4. The operator \mathcal{L}_a^ν can be rewritten in terms of the composition operator $\mathcal{C}_\psi f = f \circ \psi$ with symbol

$$\psi(z_1, z_2) = \frac{z_1 + iz_2}{\sqrt{2}}.$$

More precisely, we have

$$\mathcal{L}_a^\nu f = \left(\frac{\nu}{\pi}\right)^{-\frac{1}{4}} \mathcal{C}_\psi(\mathcal{W}_{\frac{a}{\sqrt{2}}} f).$$

Remark 4.5. The two considered operators S and \mathcal{L}_a^ν ; $a \in \mathbb{R}$, introduced above, are connected by the following

$$\mathcal{L}_a^\nu f(z_1, z_2) = S\left(\mathcal{W}_{\frac{a}{\sqrt{2}}} f\right)(-z_2, z_1)$$

for every $f \in \mathcal{F}^{2,\nu}(\mathbb{C})$.

5. Concluding remarks

Based on the classical one and two-dimensional Segal-Bargmann transforms, and a particular multiplication operator, we have been able to construct a natural integral transform $\mathcal{G}^\nu := \mathcal{B}_2 \mathcal{M}^\nu \mathcal{B}_1$, sending isometrically the standard Hilbert space $L^2(\mathbb{R}; dx)$ into the two-dimensional Bargmann-Fock space $\mathcal{F}^{2,\nu}(\mathbb{C}^2)$. It turns out that this new integral transform looks like a specific composition operator combined with the classical one-dimensional Segal-Bargmann transform. The generalization to d -complex space \mathbb{C}^d is possible only when $d = 2^n$. Indeed, we introduce the multiplication operator

$$\mathcal{M}_k : L^2(\mathbb{C}^k; e^{-\nu|z|^2} d\lambda) \longrightarrow L^2(\mathbb{C}^k; d\lambda) \simeq L^2(\mathbb{R}^{2k}; dx_1 \dots dx_{2k})$$

that is defined by

$$\mathcal{M}_k \psi(z) := e^{-\frac{\nu}{2}|z|^2} \psi(z); z = (z_1, \dots, z_k) \in \mathbb{C}^k.$$

Therefore, by induction we can construct an integral transform \mathcal{G}_n^ν mapping isometrically the standard Hilbert space $L^2(\mathbb{R}; dx)$ into the Bargmann-Fock space $\mathcal{F}(\mathbb{C}^{2^n})$. In fact is defined by $\mathcal{G}_n^\nu := \mathcal{B}_{2^n} M_{2^{n-1}} \mathcal{G}_{n-1}^\nu$ for $n > 1$ and $\mathcal{G}_1^\nu := \mathcal{G}^\nu = \mathcal{B}_2 \mathcal{M}^\nu \mathcal{B}_1$ when $n = 1$. That is

$$\mathcal{G}_n^\nu = \mathcal{B}_{2^n} M_{2^{n-1}} \mathcal{B}_{2^{n-1}} M_{2^{n-2}} \mathcal{B}_{2^{n-2}} \dots \mathcal{B}_{2^1} M_{2^0} \mathcal{B}_{2^0}.$$

Then, we claim that

$$\mathcal{G}_n^\nu \varphi = c_n \mathcal{C}_{\psi_n}(\mathcal{B}_1 \varphi)$$

where \mathcal{C}_{ψ_n} denotes the composition operator with the symbol

$$\psi_n(Z) := \frac{1}{2^{\frac{n}{2}}} \sum_{k=0}^{2^{n-1}-1} i^k (z_{2k+1} + i z_{2k+2})$$

for every $\varphi \in L^2(\mathbb{R})$ and $Z = (z_1, \dots, z_{2^n}) \in \mathbb{C}^{2^n}$. The computations hold true for $n = 1$ and $n = 2$.

Remark that the dimension 2^n that appears in this induction process is exactly the dimension of a Clifford algebra over n generators (see for example [3]). This leads to ask and investigate more about the Segal-Bargmann transform and related topics treated with Clifford Analysis methods, there has been a lot of interest in recent times especially thanks to the recent works [9, 10, 12]. Another interesting remark is related in particular to the algebra of quaternions \mathbb{H} that we can identify to \mathbb{C}^2 by considering $q = z_1 + z_2 j \mapsto (z_1, z_2)$. Thus, we think that one can find a relationship between the transform \mathcal{G}^ν and the quaternionic Segal-Bargmann transform recently introduced by the authors in the framework of the slice hyperholomorphic functions (see [4]). We claim this fact also leads to an interesting connection between the two-dimensional Bargmann space and the slice hyperholomorphic Bargmann-Fock space recently introduced in [1]. We hope to return to these ideas in forthcoming works.

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